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Relativistic paramagnetism

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Abstract. The equation of state is obtained for a degenerate system of non-interacting, relativistic, neutral fermions with spin $\frac{1}{2}$ in a uniform magnetic field. The question of arbitrarily large fields is addressed, and previous results for the ground state are extended to finite temperatures.

1. Introduction

Some attention has been given recently to the generalisation of Pauli paramagnetism to relativistic systems of non-interacting particles. The motivation for these studies stems primarily from interest in white dwarfs, and in rapidly rotating neutron stars in strong magnetic fields as models of pulsars. It has been estimated that relativistic electrons in a white dwarf can generate magnetic fields of the order of 10^7 G and higher (Lee *et al* 1969), and that in a neutron star the self-field may be of order 10^{13}–10^{14} G (Canuto and Chiu 1972). Although such fields seem enormous, they may be common in astrophysical systems and must be compared with the natural 'critical' field

$$H_c = m^2c^3/e\hbar = mc^2/2\mu_0, \tag{1}$$

where μ_0 is the appropriate magneton. For electrons $H_c \simeq 10^{13}$ G, whereas for neutrons $H_c \simeq 10^{20}$ G. Therefore in neither case is the magnetic energy more than a small fraction of particle rest energy.

Recently the ground-state zero-field magnetic susceptibility has been calculated for a completely degenerate system of relativistic charged fermions with anomalous magnetic moments (Chudnovsky 1981). For electrons in relatively small (physical) fields the calculation is straightforward and the result illustrates that paramagnetic and diamagnetic effects cannot be treated independently in this relativistic system. The author implies, however, that the resulting expression for the susceptibility is also valid for neutral fermions if one sets the charge to zero and takes the magnetic moment to be completely anomalous. A principal aim of the present discussion is to demonstrate explicitly that this conclusion is fallacious, the primary reason being that the particle spectra are entirely different in the two models. Neither neutrons, nor neutrinos if they possess a small mass and magnetic moment, follow quantised orbits in the plane normal to a uniform magnetic field.

We consider a collection of non-interacting neutral particles of rest mass m, spin $\frac{1}{2}$ and (negative) magnetic moment $\boldsymbol{\mu}_0$ in the presence of a uniform magnetic field H in the $+z$ direction. Although the particles can be thought of as contained in a volume

Ω, the infinite-volume limit will be employed consistently. One readily solves the free-particle Dirac equation with phenomenological Pauli term for the anomalous moment to obtain the positive-energy eigenvalues

$$E(\boldsymbol{p}, s)=[c^2\boldsymbol{p}^2+\mu_0^2H^2+m^2c^4+2\mu_0Hs(c^2\boldsymbol{p}_\perp^2+m^2c^4)^{1/2}]^{1/2} \tag{2}$$

where $\boldsymbol{p}_\perp$ is the component of particle momentum transverse to the field and $s=\pm 1$ is the spin-projection quantum number (Frankel *et al* 1967).

The absolute ground state of this system has been studied by Delsante and Frankel (1979), who also provide a detailed analysis of the energy spectrum (2) for arbitrary field strengths. They find rather striking behaviour of the system when $\mu_0 H > mc^2$, including a cusp in the spectrum. Further, it is found that the Fermi energy E_F vanishes as H increases without bound, and that both E_F and the total energy are independent of particle mass beyond a well defined value of the magnetic field strength.

There is ample reason to believe that these anomalous features uncovered by Delsante and Frankel are unphysical, because for field strengths of the order of H_c or greater equation (2) surely does not provide the correct spectrum. In the case of electrons this point has been discussed in some detail (Jancovici 1969, Newton 1971), where it is shown that at such superstrong magnetic field strengths the fundamental properties of electrons begin to change significantly owing to the radiative corrections from quantum electrodynamics. This assertion can be verified directly by recalling that the Dirac equation with minimal electromagnetic coupling is

$$\gamma^\mu(p_\mu - eA_\mu/c)\psi = mc\psi. \tag{3}$$

But A_μ represents the *total* four-potential, including both external fields and self-fields of the particle itself. Thus equation (3) can be written more explicitly as

$$[\gamma^\mu(p_\mu - eA_\mu^{\text{ex}}/c) - mc]\psi = (e/c)\gamma^\mu A_\mu^{\text{s}}\psi \tag{4a}$$

where in the covariant gauge

$$A_\mu^{\text{s}}(x) = e\int \mathrm{d}^4 y D(x-y)\bar{\psi}(y)\gamma_\mu\psi(y) \tag{4b}$$

and D is the electromagnetic Green function. The right-hand side of the non-linear equation (4a) contains all radiative effects for a single charged particle. In the case of a uniform magnetic field of arbitrary strength the integral has been examined carefully (Ternov *et al* 1968), and it is found that the leading-order correction to the particle energies is a very complicated function of H, particularly for strong fields. Only in the linear approximation is it possible to represent the magnetic effects as simply proportional to a constant anomalous moment μ_0. If this is true for the electron, one would expect the distortion of the neutron to be even more pronounced when $H \simeq H_c$.

In the astrophysical models of interest, however, both mc^2 and E_F are several orders of magnitude greater than $\mu_0 H$, even for fields on the order of 10^{15} G. Therefore the models of present physical interest can be described by the inequalities

$$\mu_0 H \ll (mc^2, \mu \sim E_F) \tag{5}$$

where μ is the chemical potential of the degenerate relativistic system. There is no restriction on either the absolute or relative sizes of (kT, mc^2, E_F), but the completely degenerate system is always described by $kT \ll E_F$. One can now employ the spectrum

(2) for such relatively weak magnetic fields in pure-moment systems, for which $E_F \simeq 30$ MeV, $mc^2 \simeq 10^3$ MeV.

Of course, one can extract the zero-field, ground-state magnetic susceptibility of a non-interacting neutron gas, say, from the work of Delsante and Frankel. A further goal of the present discussion, however, is to show that one can actually calculate an exact equation of state for this system under the physical restrictions (5), with temperature corrections. That is, for particles with spectrum given by equation (2) the work of Delsante and Frankel is extended to thermodynamic systems at finite temperature. Finally, the ground-state magnetic susceptibility is compared with that obtained for the electron gas in similar physical fields.

2. Degenerate equation of state

The single-particle Boltzmann partition function for a system of non-interacting pure moments is readily evaluated from equation (2) by direct summation (Frankel *et al* 1967)

$$Z_1(\beta) = \frac{4\pi\Omega}{\lambda_0^3} e^{\zeta}[(\zeta-\eta)^2 K_2(|\zeta-\eta|) + (\zeta+\eta)^2 K_2(\zeta+\eta) + \tfrac{1}{2}\pi\eta(\zeta+\eta)(K_1(\zeta+\eta)L_0(\zeta+\eta) + L_1(\zeta+\eta)K_0(\zeta+\eta)) - \tfrac{1}{2}\pi\eta(\zeta-\eta)(K_1(|\zeta-\eta|)L_0(|\zeta-\eta|) + L_1(|\zeta-\eta|)K_0(|\zeta-\eta|))] \tag{6}$$

in terms of modified Bessel functions $K_\nu(z)$ and modified Struve functions $L_\nu(z)$. The properties of these functions are described elsewhere (Abramowitz and Stegun 1972). In equation (6) the following notation has been introduced:

$$\zeta = \beta mc^2 \qquad \eta = \beta\mu_0 H \qquad \lambda_0 = \beta\hbar c \tag{7}$$

and $\beta = (kT)^{-1}$, with k Boltzmann's constant and T the absolute temperature in K. The zero of energy has been taken as zero, so that the non-relativistic limit of Z_1 is regained immediately from equation (6).

It is known that, quite generally, the quantum-statistical description of a free-particle system can be obtained directly from the single-particle partition function $Z_1(\beta)$ by means of the inverse Mellin transform representation for the grand partition function (Grandy and Rosa 1981):

$$\ln Z_G = \frac{1}{2i}\int_{c-i\infty}^{c+i\infty} t^{-1}\operatorname{cosec}(\pi t)\, e^{\beta\mu t} Z_1(\beta t)\, dt \qquad 0<c<1. \tag{8}$$

When Z_1 is given by equation (6) it is readily seen that the integral converges for all values of the parameters, and that the integrand has a branch point at the origin as well as simple poles at $t = n = \pm 1, \pm 2, \ldots$. Explicit evaluation is carried out by closing the contour and employing Cauchy's residue theorem. For example, closure to the right in a semicircle yields the fugacity expansion valid at high temperatures and low densities.

The strongly degenerate system is described by closing the contour of equation (8) to the left, as in figure 1, with appropriate indentation around the branch cut along the negative real axis. As the radius R of the quarter-circles tends to infinity the integrand will vanish on those portions of the contour labelled BC and DA,

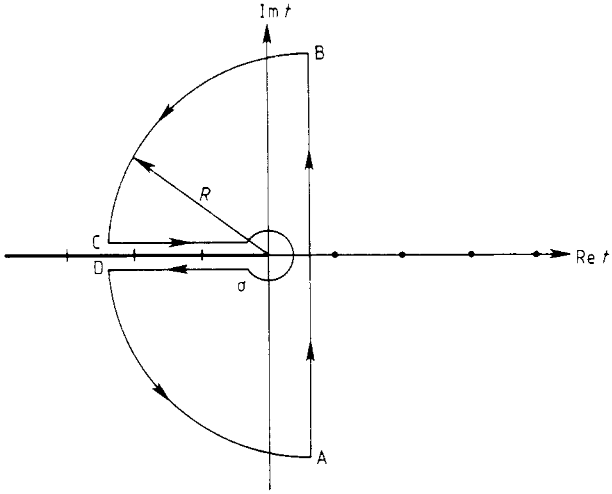

Figure 1. Contour employed in evaluating the equation of state for the degenerate system.

provided that $\eta < \beta\mu$, $\eta < \beta\mu + 2\zeta$. These are just the conditions implied by equation (5). The integral for $\ln Z_G$ therefore reduces to the negative of that portion on σ from C to D along the branch cut and encircling the origin in the negative direction, in the limit $R \to \infty$.

Substitution of equation (6) into equation (8) yields a sum of six separate integrals taken around the contour σ, each of which can be evaluated by means of techniques similar to those employed in the zero-field case (Grandy and Rosa 1981). Indeed, the first two integrals, corresponding to the terms in $K_2(z)$ in equation (6), have precisely the same form as that encountered for $H = 0$, and so are identified immediately. Introduce simple generalisations of the zero-field parameters

$$\alpha_\pm = \alpha\zeta/(\zeta \pm \eta) \xrightarrow[\eta \to 0]{} \alpha = 1 + \mu/mc^2 \tag{9a}$$

$$x_\pm = \alpha_\pm^2 - 1 \xrightarrow[\eta \to 0]{} x = \alpha^2 - 1 \tag{9b}$$

where $x = p_0/mc$ and p_0 is the solution of

$$\mu = (c^2 p_0^2 + m^2 c^4)^{1/2} - mc^2. \tag{9c}$$

The correct field-dependent Fermi energy, of course, is determined by writing $\mu = E(\boldsymbol{p}, s) - mc^2$, in which there can be no ambiguity because the chemical potentials for spin-up and spin-down particles must be equal in thermal equilibrium.

If we write the pressure as

$$\begin{aligned} P &= (\beta\Omega)^{-1} \ln Z_G = P_1 + P_2 \\ P_1 &= P_1(+) + P_1(-) \qquad P_2 = P_2(+) + P_2(-) \end{aligned} \tag{10}$$

then the two integrals containing K_2 yield the contribution

$$\begin{aligned} P_1(\pm) = \frac{\pi}{6}\frac{mc^2}{\lambda_c^3}[f(x_\pm) &+ 4\pi^2\zeta^{-2}x_\pm(x_\pm^2+1)^{1/2}(\alpha/\alpha_\pm)^2 \\ &+ \tfrac{7}{15}\pi^4\zeta^{-4}x_\pm^{-3}(2x_\pm^2-1)(x_\pm^2+1)^{1/2} + \dots] \end{aligned} \tag{11}$$

where $\lambda_c = h/mc$ and

$$f(x) = x(2x^2-3)(1+x^2)^{1/2} + 3\sinh^{-1} x \tag{12}$$

is the characteristic function of the zero-field theory. As in that case (Chandrasekhar 1957) the temperature expansion of equation (11) is indeed in terms of small quantities, because

$$\frac{x_\pm(x_\pm^2+1)^{1/2}(\alpha/\alpha_\pm)^2}{\zeta^2 f(x_\pm)(\alpha/\alpha_\pm)^4} \sim \frac{1}{\alpha^2\zeta^2} \ll 1, \tag{13}$$

which implies the smallness of succeeding terms.

The remaining four integrals arising from equations (6) and (8) contain Struve functions and vanish as $H \to 0$. They can be evaluated formally in a manner similar to that employed above and we find that

$$\begin{aligned} P_2(\pm) = \pm(2\pi^3/\lambda_c^3\zeta^3)(\alpha\zeta/\alpha_\pm)\eta[(\alpha\zeta/\pi)^2 + \tfrac{1}{3} + (\tfrac{2}{9}\pi^2)(\alpha\zeta/\alpha_\pm)^2 + (2\gamma-1)(\alpha\zeta/\alpha_\pm)^2/2\pi^2 \\ - 3(\alpha\zeta/3\pi\alpha_\pm)^2\ln(2\alpha_\pm) - (2\gamma/3\pi^2)(\alpha\zeta/\alpha_\pm)^2 \\ + (\alpha^2\zeta^2/\pi)J_1(\alpha_\pm) + \alpha\zeta J_2(\alpha_\pm)] \end{aligned} \tag{14}$$

where $\gamma = 0.577\,21\ldots$ is the Euler–Mascheroni constant. The functions J_1 and J_2 are defined as

$$J_1(\alpha_\pm) = \int_0^\infty \mathrm{d}y \ln y\, \mathrm{e}^{-y} y^{-2}[G(y/\alpha_\pm)(1+3/y) - (2/\pi\alpha_\pm)I_1(y/\alpha_\pm)] \tag{15}$$

$$J_2(\alpha_\pm) = \int_0^\infty \mathrm{d}y\, \mathrm{e}^{-y} y^{-2} G(y/\alpha_\pm)[(\pi/6\alpha\zeta)y + (7\pi^3/360\alpha^3\zeta^3)y^3 + \ldots] \tag{16}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind and

$$\begin{aligned} G(x) &= I_1(x)L_0(x) - I_0(x)L_1(x) \\ &= \frac{2}{\pi x}\int_0^x tI_1(t)\,\mathrm{d}t. \end{aligned} \tag{17}$$

Remarkably, the integrals J_1 and J_2 can be evaluated exactly in terms of generalised hypergeometric functions, and then in terms of elementary transcendental functions. This evaluation is outlined in the appendix. Thus, with equations (10), (11) and (14) we have obtained an exact equation of state for the degenerate system, with temperature corrections. All of the thermodynamic functions can now be found by direct differentiation.

3. Ground-state magnetisation

At $T = 0$ the pressure of the completely degenerate system is given by

$$P = P(+) + P(-) \tag{18a}$$

$$\begin{aligned} P(\pm) = \frac{1}{48\pi^2}\frac{(mc^2)^4}{(\hbar c)^3} f(x_\pm)(\alpha/\alpha_\pm)^4 \pm \frac{1}{2\pi^2}\frac{(\mu_0 H)(mc^2)^3}{(\hbar c)^3} \\ \times(\alpha/\alpha_\pm)^3(\tfrac{1}{3}x_\pm\alpha_\pm - \tfrac{1}{6}\sinh^{-1}(x_\pm) + \tfrac{1}{6}\alpha_\pm^3\cos^{-1}(x_\pm/\alpha_\pm)) \end{aligned} \tag{18b}$$

where $f(x_\pm)$ is defined in equation (12) and equation (A.12) has been used for $J_1(\alpha_\pm)$. This is the exact equation of state in this limit, with no approximations, as long as $\mu_0 H$ is less than both mc^2 and E_F.

Owing to the inequalities (5), it is useful at this point to examine the (relatively) weak-field case, in which only leading-order contributions in $\mu_0 H$ are retained. Careful approximation of all the quantities in equations (18) yields

$$(\pi^2\hbar^3c^3/m^4c^8)P \simeq \tfrac{1}{24}f(x) + \tfrac{1}{4}(\eta^2/\zeta^2)[x(1+x^2)^{1/2} + \sinh^{-1}(x)] \tag{19}$$

where x is the zero-field quantity defined by equation (9c). One readily computes the number density $n = N/\Omega$ in this approximation:

$$\begin{aligned} n &= \partial P/\partial\mu \\ &\simeq \frac{(mc^2)^3}{3\pi^2(\hbar c)^3}[x^3 + (\eta/\zeta)^2 3(x^2+2)/2x]. \end{aligned} \tag{20}$$

Both the number density and Fermi energy are thus given to this order by the zero-field results. Hence,

$$x \xrightarrow[T\to 0]{} x_F = p_F/mc \tag{21}$$

which reduces to $(2E_F/mc^2)^{1/2}$ in the non-relativistic limit. That is, the quantity in brackets in equation (19) reduces to $\frac{1}{2}x_F$, which is indeed the correct limiting result.

In order to facilitate comparison with other results it is convenient to introduce the additional notation

$$a = |\mu_0 H| \qquad E_0 = mc^2 \tag{22}$$

$$\varepsilon = \mu + E_0 \simeq E_F + E_0 = [E_0^2 + (\hbar c)^2(3\pi^2 n)^{2/3}]^{1/2}. \tag{23}$$

With this notation, and the logarithmic representation of $\sinh^{-1}(x)$, equation (19) can be rewritten as

$$\begin{aligned} (\pi^2\hbar^3c^3/E_0^4)P \simeq{}& \tfrac{1}{24}\left((\varepsilon/E_0^4)(\varepsilon^2 - E_0^2)^{1/2}(2\varepsilon^2 - 5E_0^2) + 3\ln\left(\frac{\varepsilon + (\varepsilon^2 - E_0^2)^{1/2}}{\varepsilon_0}\right)\right) \\ &+ \frac{a^2}{4E_0^4}\left(\varepsilon(\varepsilon^2 - E_0^2)^{1/2} + E_0^2\ln\left(\frac{\varepsilon + (\varepsilon^2 - E_0^2)^{1/2}}{E_0}\right)\right). \end{aligned} \tag{24}$$

If one now calculates the energy density, E/Ω, the earlier result is regained (Delsante and Frankel 1979), with the difference that these authors took the zero of energy to be E_0.

The magnetisation is obtained directly from equation (19), $M = \partial P/\partial H$, so that the ground-state, zero-field magnetic susceptibility is

$$\begin{aligned} \chi &= \partial M/\partial H \\ &\simeq \frac{E_0^2\mu_0^2}{2\pi^2(\hbar c)^3}\left(\frac{\varepsilon}{E_0^2}(\varepsilon^2 - E_0^2)^{1/2} + \ln\left(\frac{\varepsilon + (\varepsilon^2 - E_0^2)^{1/2}}{E_0}\right)\right). \end{aligned} \tag{25}$$

This is precisely what one would obtain from the work of Delsante and Frankel, but is here deduced from a more general description of the system.

It is also instructive to introduce the Fermi velocity $v_F = cx_F$ and rewrite equation (25) as

$$\chi \simeq \frac{E_0^2\mu_0^2}{2\pi^2(\hbar c)^3}[(v_F/c)(1+v_F^2/c^2)^{1/2}+\ln[(v_F/c)+(1+v_F^2/c^2)^{1/2}]]$$
$$\xrightarrow[v_F \ll c]{} 3n\mu_0^2/2E_F, \tag{26}$$

exhibiting the non-relativistic limit of Pauli paramagnetism.

Unlike the electron gas (Chudnovsky 1981), the paramagnetic pure-moment system is completely stable, even as $v_F \to c$. Moreover, there is essentially no relation between (26) and the corresponding result for the electron gas, for just the reason emphasised by Chudnovsky: in the relativistic system the paramagnetic and diamagnetic effects are intertwined such that they are not easily separated, either physically or mathematically.

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Appendix. Evaluation of the integrals J_1 and J_2

The integrals defined in equations (15) and (16) of the text are evaluated by first noting that the integrands can be expanded in uniformly convergent power series. One then integrates term-by-term, after which it is possible to identify the resulting series as generalised hypergeometric functions

$$_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \tag{A.1}$$

where $(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)$. Specifically,

$$J_1(\alpha_\pm) = -\gamma/3\pi\alpha_\pm^2 + (40\pi\alpha_\pm^4)^{-1}{}_4F_3(1, 1, \tfrac{3}{2}, \tfrac{5}{2}; 2, 3, \tfrac{7}{2}; \alpha_\pm^{-2}) \tag{A.2}$$

$$J_2(\alpha_\pm) = \alpha_\pm^{-2}[(6\alpha\zeta)^{-1}{}_3F_2(\tfrac{3}{2}, \tfrac{3}{2}, 1; 2, \tfrac{5}{2}; \alpha_\pm^{-2}) + (7\pi^2/180)(\alpha\zeta)^{-3}{}_1F_0(\tfrac{3}{2}; 0; \alpha_\pm^{-2}) + \ldots], \tag{A.3}$$

all of which converge within the unit circle. If $\mu_0 H < E_F$, then $\alpha_\pm^{-2} < 1$ always.

It is a straightforward matter to verify by direct expansion that, in J_2,

$$_1F_0(\tfrac{3}{2}; 0; z) = (1-z)^{-3/2} \tag{A.4}$$

$$_3F_2(\tfrac{3}{2}, \tfrac{3}{2}, 1; 2, \tfrac{5}{2}; z) = 6[1-z+(1-z)^{1/2}]^{-1} - 6z^{-3/2}(1-z)^{1/2}[z^{1/2}-(1-z)^{1/2}\sin^{-1}(z^{1/2})]. \tag{A.5}$$

These functions only contribute at finite temperatures.

The evaluation of ${}_4F_3$ is less direct, beginning with the contiguous function relation (Rainville 1945, 1960)

$$ {}_4F_3(1, 1, \tfrac{3}{2}, \tfrac{5}{2}; 2, 3, \tfrac{7}{2}; z) = \tfrac{5}{3}\,{}_3F_2(1, 1, \tfrac{3}{2}; 2, 3; z) - \tfrac{2}{3}\,{}_3F_2(1, \tfrac{3}{2}, \tfrac{5}{2}; 3, \tfrac{7}{2}; z). \tag{A.6} $$

Each of the functions ${}_3F_2$ can be evaluated by means of its integral representation (Rainville 1960), which are almost identical:

$$ {}_3F_2(1, 1, \tfrac{3}{2}; 2, 3; z) = \int_0^1 {}_2F_1(1, \tfrac{3}{2}; 3; zt)\, \mathrm{d}t \tag{A.7} $$

$$ {}_3F_2(1, \tfrac{3}{2}, \tfrac{5}{2}; 3, \tfrac{7}{2}; z) = \tfrac{5}{2} \int_0^1 t^{3/2}\,{}_2F_1(1, \tfrac{3}{2}; 3; zt)\, \mathrm{d}t. \tag{A.8} $$

Again by direct expansion, one verifies that

$$ {}_2F_1(1, \tfrac{3}{2}; 3; y) = 4[1 + (1-y)^{1/2}]^{-2} \tag{A.9} $$

and straightforward integration yields

$$ {}_3F_2(1, 1, \tfrac{3}{2}; 2, 3; z) = 8z^{-1}[\tfrac{1}{2} - z^{-1} + z^{-1}(1-z)^{1/2} + \ln 2 - \ln[1 + (1-z)^{1/2}]] \tag{A.10} $$

$$ {}_3F_2(1, \tfrac{3}{2}, \tfrac{5}{2}; 3, \tfrac{7}{2}; z) = 20z^{-5/2}[2z^{1/2} - z^{1/2}(1-z)^{1/2} - \tfrac{1}{3}z^{3/2} - \cos^{-1}(1-z)^{1/2}]. \tag{A.11} $$

With substitution of equations (A.6), (A.10) and (A.11) into equation (A.2) we obtain the simple expression

$$ \begin{aligned} J_1(\alpha_\pm) = -\gamma(3\pi\alpha_\pm^2)^{-1} + (3\pi\alpha_\pm^4)^{-1}(\tfrac{5}{6}\alpha_\pm^2 - 3\alpha_\pm^4 + 2x_\pm\alpha_\pm^3 \\ + \alpha_\pm^2 \ln(2\alpha_\pm) - \alpha_\pm^2 \sinh^{-1}(x_\pm) + \alpha_\pm^5 \cos^{-1}(x_\pm/\alpha_\pm)). \end{aligned} \tag{A.12} $$

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